Optimal Decision Making and the Value of Information in a Time-Dependent Version of the Cost–Loss Ratio Situation

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(Manuscript received 8 May 1989, in final form 14 October 1989)

ABSTRACT

A time-dependent version of the cost–loss ratio situation is described and the optimal use and economic value of meteorological information are investigated in this decision-making problem. The time-dependent situation is motivated by a decision maker who contemplates postponing the protect/do not protect decision in anticipation of obtaining more accurate forecasts at some later time (i.e., shorter lead time), but who also recognizes that the cost of protection will increase as lead time decreases. Imperfect categorical forecasts, calibrated according to past performance, constitute the information of primary interest. Optimal decisions are based on minimizing expected expense and the value of information is measured relative to the expected expense associated with climatological information.

Accuracy and cost of protection are modeled as exponentially decreasing functions of lead time, and time-dependent expressions for expected expense and value of information are derived. An optimal lead time is identified that corresponds to the time at which the expected expense associated with imperfect forecasts attains its minimum value. The effects of the values of the parameters in the accuracy and cost-of-protection models on expected expense, optimal lead time, and forecast value are examined. Moreover, the optimal lead time is shown to differ in some cases from the lead time at which the economic value of imperfect forecasts is maximized. Numerical examples are presented to illustrate the various results. The implications of these results are discussed and some possible extensions of this work are suggested.

1. Introduction

Prescriptive studies of applications of weather and climate information, based on models of decision-making problems, can provide valuable insights into the rational use and economic benefits of meteorological forecasts. Moreover, although it is obviously desirable to investigate forecast use and value in the context of real-world situations, consideration of prototype weather/climate-information-sensitive problems can also yield important results. The so-called "cost–loss ratio situation" is a simple prototype decision-making problem that has been used extensively within the meteorological community as a means of investigating the optimal use and economic value of weather and climate forecasts.

The basic cost–loss ratio situation is a static (that is, "one-shot") decision-making problem in which a decision maker must decide *at a particular time* whether or not to protect an activity or operation against adverse weather in the face of uncertainty as to whether or not such weather conditions will actually occur. If protective action is taken, a cost of protection is incurred; however, an even larger loss is experienced if protective action is not taken and adverse weather occurs. Since this situation is (by design) weather/climate-information-sensitive, it provides a potentially useful framework within which to investigate the use and value of meteorological forecasts. Studies involving a static, fixed-time version of the two-action, two-event cost–loss ratio situation have been undertaken by Thompson (1952), Thompson (1962), and Murphy (1977) among others.

In recent years the basic cost–loss ratio situation has been extended in a variety of ways. For example, Murphy (1985) described an *N*-action, *N*-event extension of the static, fixed-time model. Moreover, dynamic or sequential models have been formulated to analyze interrelated repetitive decisions in the context of the basic situation (e.g., Epstein and Murphy 1988; Murphy et al. 1985). Moreover, considerable attention has been devoted to the investigation of the relationship between forecast quality and forecast value in the context of both static and dynamic versions of this problem (e.g., Katz and Murphy 1987, 1990; Murphy et al. 1985).

The purpose of this paper is to investigate a *time-dependent version* of the cost–loss ratio situation. In this situation, only a single decision (to protect or not protect) is made, but the decision maker may choose to postpone the decision until forecasts of greater accuracy (associated with a shorter lead time) become available. The penalty for postponing the decision

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manifests itself in terms of an increased cost of protection. Thus a tradeoff may exist between waiting for more accurate forecasts—which generally would lead to less "expensive" decisions—and possibly incurring a greater expense due to more costly protective measures. This paper investigates the nature of this tradeoff both in general and for specific forms of accuracy/lead-time and cost-of-protection/lead-time functions and explores optimal decisions and the value of information in this context.

The time-dependent cost–loss ratio situation is described in section 2. This section also presents general expressions for the expected expense and expected value of imperfect forecasts. Section 3 includes specific expressions and results for the situation in which the relevant time-dependent quantities are modeled using exponential functions. These results are illustrated by means of several numerical examples in section 4. Section 5 contains a discussion of the results and some concluding remarks.

2. Time-dependent situation: background and basic considerations

a. Fixed-time situation

The basic cost–loss ratio situation is a static, fixed-time, decision-making problem involving two actions—protect ($a = 1$) and do not protect ($a = 0$)—and two events—adverse weather ($x = 1$) and no adverse weather ($x = 0$). If the decision maker takes protective action ($a = 1$), a cost of protection C is incurred (regardless of whether $x = 1$ or $x = 0$). On the other hand, if protective action is not taken ($a = 0$) and adverse weather occurs ($x = 1$) the decision maker experiences a loss L. Finally, if protective action is not taken ($a = 0$) and adverse weather does not occur ($x = 0$), no expense (cost or loss) is incurred. In order to consider situations with nontrivial solutions, we assume throughout this paper that $0 < C < L$.

In the absence of forecasts, it is assumed that the protect/do not protect decision is made on the basis of climatological information. This information consists solely of the climatological (or prior) probability of adverse weather π, where $\pi = P(x = 1)[P(x = 0) = 1 - \pi]$. The forecasts of interest here are categorical forecasts of adverse weather ($z = 1$) or no adverse weather ($z = 0$). After calibration, these imperfect categorical forecasts are represented by the conditional (or posterior) probabilities p_1 and p_0, where $p_1 = P(x = 1|z = 1)$ and $p_0 = P(x = 1|z = 0)[P(x = 0|z = 1) = 1 - p_1$ and $P(x = 0|z = 0) = 1 - p_0]$. Thus, calibrated categorical forecasts can also be viewed as primitive (that is, two-valued) probabilistic forecasts, with probability values p_1 and p_0. Without loss of generality, it can be assumed that the climatological and conditional probabilities satisfy the ordering $0 \leqslant p_0 \leqslant \pi \leqslant p_1 \leqslant 1$. Finally, the marginal (or predictive)

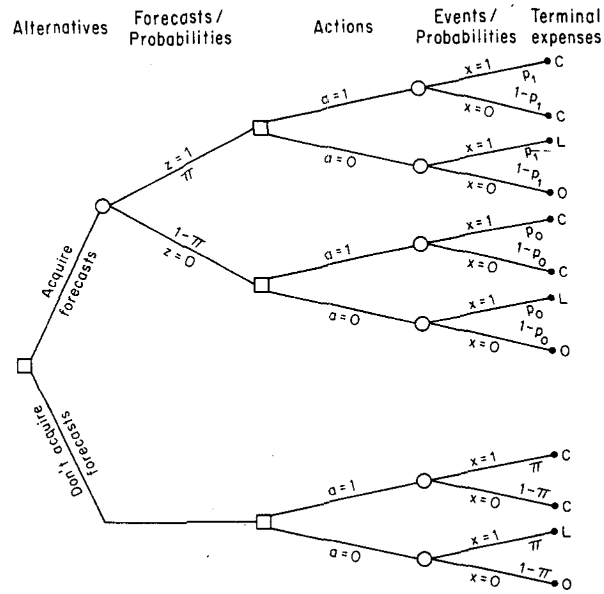

FIG. 1. Decision tree for basic fixed-time cost–loss ratio situation (recall that $p_z = \pi$; see text for additional details).

probability of a forecast of adverse weather is denoted by p_z, where $p_z = P(z = 1)[P(z = 0) = 1 - p_z]$.

Consistency between the climatological, conditional, and predictive probabilities requires that

$$\pi = p_z p_1 + (1 - p_z)p_0. \tag{1}$$

Moreover, to simplify matters further in this paper, we will assume that $p_z = \pi$. That is, it is assumed that the probability of a forecast of adverse weather is equal to the climatological probability of adverse weather. This assumption implies, from (1), that

$$p_0 = \pi(1 - p_1)/(1 - \pi). \tag{2}$$

Thus, under the assumption that π is known, the probability p_1 ($\pi \leqslant p_1 \leqslant 1$) completely determines the characteristics of the imperfect forecasts. In particular, climatological and perfect information are the special, limiting cases of the forecasts in which $p_1 = \pi$ and $p_1 = 1$, respectively.

The static, fixed-time model of the cost–loss ratio situation is depicted in the form of a decision tree in Fig. 1. This tree contains the relevant actions, events, (terminal) expenses, forecasts, and probabilities (climatological, conditional, and predictive). The initial decision, on the part of the decision maker, relates to whether or not to "acquire" the forecasts, and this decision is assumed to be based on the expected expenses associated with these two alternatives (see section 2c). Under the assumption that the cost of acquiring the forecasts is negligible, the difference between these expected expenses represents the value of the forecasts (see section 2d).

b. Time-dependent situation

To motivate the formulation of a time-dependent version of the basic cost–loss ratio problem, consider a situation in which a decision maker contemplates postponing the protect/do not protect decision in anticipation of obtaining more accurate forecasts at some later time (that is, shorter lead time). In this regard, almost all studies of forecast accuracy as a function of lead time have shown that the former is a decreasing function of the latter (e.g., Murphy and Sabin 1986; Sanders 1986). On the other hand, the cost of protection would be expected to increase as lead time decreases because additional resources (e.g., equipment and/or personnel) generally would be required to protect the activity or operation in the remaining time available. Thus a potential tradeoff may exist as lead time decreases between a decrease in expected expense associated with forecasts of greater accuracy and an increase in expected expense arising from a greater cost of protection. It seems reasonable to assume that the loss L would *not* be time dependent.

This time-dependent decision-making problem is depicted schematically in the form of a decision tree in Fig. 2. Here, lead time (the time between the protect/do not protect decision and the start of the period during which the activity actually takes place) is denoted by t and is represented by a discrete set of integer values ($t = 1, \cdots, T$). The decision maker is assumed to choose a particular time t at which to acquire the forecasts or to decide not to acquire the forecasts; in the latter case, the protect/do not protect decision is based on climatological information (which is *not* time dependent). Note that both the cost of protection C and the parameter p_1—the measure of forecast accuracy (see section 2a)—are considered to be functions of lead time t.

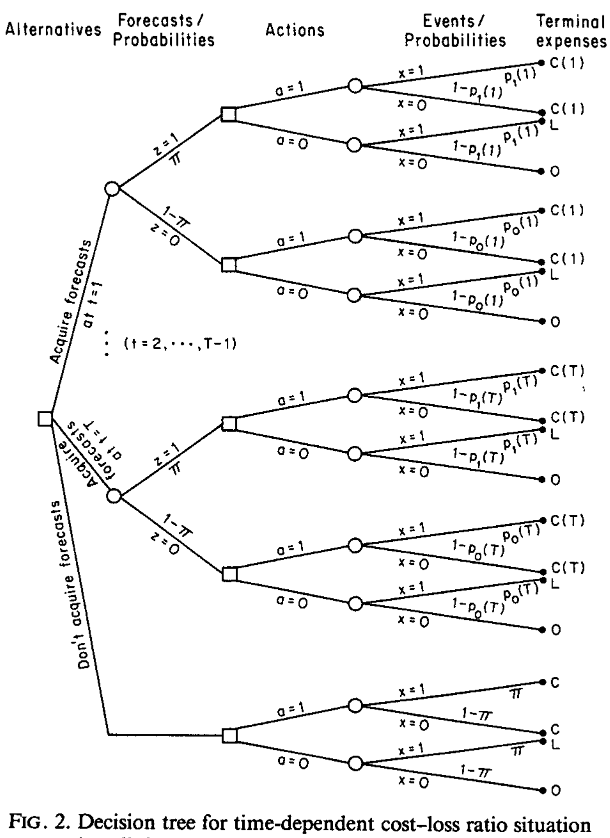

FIG. 2. Decision tree for time-dependent cost-loss ratio situation (recall that $p_z = \pi$; see text for additional details).

More specifically, both $C = C(t)$ and $p_1 = p_1(t)$ are assumed to be continuous, monotonically decreasing functions of t in this paper. In the case of the cost of protection, $C(t)$ is assumed to be relatively large for short lead times [$C(0) < L$] and relatively small for long lead times [$C(\infty) > 0$]. Moreover, the accuracy of the imperfect forecasts is assumed to approach the accuracy of perfect forecasts [that is, $p_1(t) \rightarrow 1$] as t approaches zero and to approach the accuracy of climatological information [that is, $p_1(t) \rightarrow \pi$] as t approaches infinity. Specific functional forms for $C(t)$ and $p_1(t)$ will be introduced in section 3a.

c. Expected expenses and optimal decisions

It will be assumed throughout this paper that the decision criterion is the minimization of expected expense. That is, the decision maker is assumed to choose the action—protect ($a = 1$) or do not protect ($a = 0$)—for which the expected expense is a minimum, where the expected expense of an action is the probability-weighted average of the expenses (costs and/or losses) associated with that action. It is also assumed that this same criterion is applied to the decision related to the acquisition of the forecasts. Minimizing expected expense is equivalent to maximizing expected utility under the assumption that the decision maker's utilities are linearly related to the expenses (Winkler and Murphy 1985).

1) CLIMATOLOGICAL INFORMATION

Let $E_C(a = 1)$ and $E_C(a = 0)$ denote the expected expenses associated with taking and not taking protective action when these decisions are based solely on climatological information. In the basic fixed-time situation, $E_C(a = 1) = \pi C + (1 - \pi)C = C$ and $E_C(a = 0) = \pi L + (1 - \pi)0 = \pi L$. Thus, the decision maker will choose $a = 1$ (protect) if $\pi > C/L$ and $a = 0$ (do not protect) if $\pi < C/L$. Moreover, if the minimum expected expense (incurred by the decision maker) in this case is denoted by E_C, then

$$E_C = \min(C, \pi L). \tag{3}$$

It will be convenient to rescale this minimum expected expense by dividing by the loss L. Thus, if $E'_C = E_C/L$, then

$$E'_C = \min(C', \pi) \tag{4}$$

[from (3)], where $C' = C/L$.

In the time-dependent situation, $C' = C'(t)$ [$=C(t)/L$] and $E'_C = E'_C(t)$, where

$$E'_C(t) = \min[C'(t), \pi] \tag{5}$$

[from (4)]. Since $C'(t)$ is assumed to be a (monotonically) decreasing function of t, it is evident from (5) that three distinct cases can be identified: 1) case 1, $\pi < C'(t)$ for all values of t; 2) case 2, $\pi < C'(t)$ for small values of t and $\pi > C'(t)$ for large values of t; and 3) case 3, $\pi > C'(t)$ for all values of t. In case 1 the decision maker will never protect and $E'_C(t) = \pi$ (a constant) for all t. On the other hand, in case 2 a value of t exists for which $\pi = C'(t)$. If this critical value of t is denoted by t^*, then the decision maker will not protect for $t < t^*$ with $E'_C(t) = \pi$, whereas he/she will protect for $t > t^*$, with $E'_C(t) = C'(t)$ (a decreasing function of t). Thus, t^* plays the role in the time-dependent situation that the cost-loss ratio (C/L) plays in the fixed-time situation. Finally, in case 3 the decision maker will always protect and $E'_C(t) = C'(t)$ for all t. Since the value of imperfect forecasts depends on $E'_C(t)$ (see section 2d), these three cases will be of interest in subsequent sections of the paper.

2) IMPERFECT FORECASTS

If we let E_F denote the expected expense associated with the forecasts in the fixed-time situation, then

$$E_F = \pi C + (1 - \pi)p_0 L \tag{6}$$

(e.g., Katz and Murphy 1987) or, from (2),

$$E_F = \pi C + \pi(1 - p_1)L. \tag{7}$$

Strictly speaking, the expressions for E_F in (6) and (7) are valid only when $p_0 < C/L < p_1$. However, when $C/L < p_0 < p_1$ or $p_0 < p_1 < C/L$, the expression for E_F is identical to E_C in (3) (under these conditions, π, p_0, and p_1 all lead to the same optimal actions). Since we are primarily concerned in this paper with situations in which a positive difference exists between E_C and E_F (that is, situations in which the value of information is nonzero), attention is focused here on the case in which $p_0 < C/L < p_1$.

The first term on the right-hand side (rhs) of (7) relates to forecasts of adverse weather ($z = 1$) and taking protection action ($a = 1$), whereas the second term on the rhs of (7) relates to forecasts of no adverse weather ($z = 0$) and not taking protective action ($a = 0$). As in the case of climatological information, it will be convenient to rescale E_F in (7) by dividing by the loss L. Thus, if $E'_F = E_F/L$, then from (7),

$$E'_F = \pi C' + \pi(1 - p_1). \tag{8}$$

In the time-dependent situation, $C' = C'(t)$, $p_1 = p_1(t)$, and $E'_F = E'_F(t)$, where

$$E'_F(t) = \pi[1 + C'(t) - p_1(t)] \tag{9}$$

[from (8)]. Since both $C'(t)$ and $p_1(t)$ are decreasing functions of t, the behavior of $E'_F(t)$ in (9) depends on the specific forms of these time-dependent functions. In general, however, the decision maker wants to make his/her decision on the basis of the imperfect forecasts at the lead time t for which $E'_F(t)$ is a minimum. If we denote this lead time by $t_{\min}$, then it is evident from (9) that such a minimum exists only if $\partial C'(t)/\partial t|_{t=t_{\min}} = \partial p_1(t)/\partial t|_{t=t_{\min}}$ [obtained by setting $\partial E'_F(t)/\partial t = 0$] and $\partial^2 C'(t)/\partial t^2|_{t=t_{\min}} > \partial^2 p_1(t)/\partial t^2|_{t=t_{\min}}$. The expression for $t_{\min}$, an important parameter in this study, obviously depends on the forms of the functions $C'(t)$ and $p_1(t)$.

d. *Value of information*

Here the phrase "value of information" refers to the economic value of the imperfect forecasts, which is defined as the difference between the expected expense associated with climatological information and the expected expense associated with the forecasts. Thus, if V'_F denotes the value of these forecasts (standardized by dividing by L) in the fixed-time situation, then

$$V'_F = E'_C - E'_F, \tag{10}$$

or, from (4) and (8),

$$V'_F = \min(C', \pi) - \pi C' - \pi(1 - p_1). \tag{11}$$

In the *ex ante,* decision-analytic approach to the value of information taken in this paper, $V'_F \geqslant 0$ (e.g., see Katz and Murphy 1987).

In the time-dependent situation, $C' = C'(t)$, $p_1 = p_1(t)$, and $V'_F = V'_F(t)$, where

$$V'_F(t) = \min[C'(t), \pi] - \pi[1 + C'(t) - p_1(t)] \tag{12}$$

[from (11)]. As in the case of $E'_F(t)$, the behavior of $V'_F(t)$ in (12) depends on the specific forms of the functions $C'(t)$ and $p_1(t)$. In this regard, however, the lead time t at which $V'_F(t)$ attains its maximum value is of some interest (from the point of view of the forecaster). We will denote this lead time by $t_{\max}$, which can be determined by differentiating $V'_F(t)$ with respect to t and setting this partial derivative equal to zero [moreover, the second derivative of $V'_F(t)$ must be less than zero at $t_{\max}$ for the maximum to exist].

Examination of $V'_F(t)$ in (12) reveals that, when $\pi < C'(t)$ (case 1 and, when this condition holds, case 2), $V'_F(t)$ is linearly related to $E'_F(t)$ [since $E'_C(t) = \pi$, a constant]. As a result, $t_{\max} = t_{\min}$ in this situation; that is, the expected expense associated with imperfect forecasts and the expected value of these forecasts are minimized and maximized, respectively, at the same lead time. On the other hand, when $\pi > C'(t)$ (case 2, when this condition holds, and case 3), $V'_F(t)$ is no longer linearly related to $E'_F(t)$ [since $E'_C(t) = C'(t)$, a time-dependent function], with the result that $t_{\max}$

$\neq t_{\min}$ (in general). In this latter situation, the lead time that minimizes the expected expense associated with the forecasts will not be the lead time that maximizes the expected value of the forecasts. These considerations will be discussed for the specific functional forms of the parameters $C'(t)$ and $p_1(t)$ in section 3c.

3. Time-dependent situation: specific models

a. Exponential accuracy and cost-of-protection models

It is assumed here that both the accuracy parameter $p_1(t)$ and the cost-of-protection parameter $C'(t)$ decrease exponentially as lead time t increases. In the case of $p_1(t)$, some evidence exists (e.g., see Sanders 1986; Weingärtner 1987) to support this assumption. Specifically, we assume that $p_1(t)$ is an exponentially decreasing function of t of the following form:

$$p_1(t) = (1 - \pi)e^{-At} + \pi, \qquad (13)$$

where A (>0) is a parameter to be specified. This particular exponential function has been chosen so that $p_1(t) = 1$ when $t = 0$ (zero lead time) and $p_1(t) \rightarrow \pi$ when $t \rightarrow \infty$ (infinite lead time). As indicated in section 2, the imperfect forecasts approach the accuracy of perfect information for very short lead times and approach the accuracy of climatological information for very long lead times. The parameter A determines the rate at which $p_1(t)$ increases (decreases) as t decreases (increases). It can be interpreted in terms of the e-folding time (t_e) for the "standardized" version of this model; that is, the time at which $p'_1(t)\ \{=[p_1(t) - \pi]/(1 - \pi)\}$ decreases to $1/e$ of its initial value (that is, its value at $t = 0$). In this case $t_e = 1/A$.

The behavior of $p_1(t)$ for selected values of A is depicted in Fig. 3 for values of t on the closed interval [0, 10], when $\pi = 0.2$. As described by $p_1(t)$, the rate of decrease in forecast accuracy as t increases is greater for larger values of A than for smaller values of A. For $t = 3$, $p_1(t) \approx 0.89, 0.79, 0.58,$ and 0.38 when $A = 0.05$, 0.10, 0.25, and 0.50, respectively. If the units of t are taken to be days, then these values of A correspond to e-folding times of 20, 10, 4, and 2 days, respectively. Larger values of A such as 0.25 and 0.50 would appear to be consistent with the current state-of-the-art of day-to-day weather forecasting.

In the case of $C'(t)$, we assume that this parameter is an exponentially decreasing function of lead time t of the following form:

$$C'(t) = (C'_x - C'_n)e^{-Bt} + C'_n, \qquad (14)$$

where B(>0), C'_x (>0), and C'_n (>0) are parameters to be specified. Moreover, it is assumed that $0 < C'_n < C'_x < 1$. This particular exponential function was chosen so that $C'(t) = C'_x$ when $t = 0$ and $C'(t) \rightarrow C'_n$ when $t \rightarrow \infty$. Thus, the cost of protection approaches a constant value C'_x (a maximum cost, C_x/L, which is less than unity) for very short lead times and approaches a constant value C'_n (a minimum cost, C_n/L, which is greater than zero) for very long lead times. The parameter B determines the rate at which $C'(t)$ increases (decreases) as t decreases (increases), with $t_e = 1/B$ for this model.

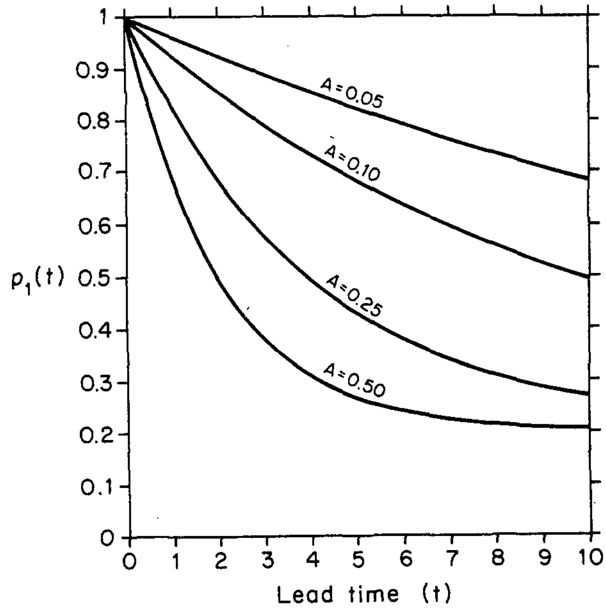

FIG. 3. The behavior of the exponential model of the accuracy parameter $p_1(t)$ for selected values of the parameter A, with $\pi = 0.2$ [see (13)].

The behavior of $C'(t)$ for selected values of the parameter B is depicted in Fig. 4 for values of t on the closed interval [0, 10], when $C'_x = 0.85$ and $C'_n = 0.05$. As in the case of $p_1(t)$, the rate of decrease in $C'(t)$ as t increases is greater for larger values of B than for smaller values of B. Since the functions $C'(t)$ and $p_1(t)$ are defined in an analogous manner, it is not surprising that they exhibit similar behavior (including identical e-folding times).

b. Expected expenses and optimal actions

1) CLIMATOLOGICAL INFORMATION

For the exponential model of the cost parameter $C'(t)$ defined in (14), $E'_C(t)$ in (5) becomes

$$E'_C(t) = \min[(C'_x - C'_n)e^{-Bt} + C'_n, \pi]. \qquad (15)$$

In terms of the parameters of this model, the three cases identified in section 2c can now be described as follows: 1) case 1, $\pi < C'_n < C'_x$; 2) case 2, $C'_n < \pi < C'_x$; and 3) case 3, $C'_n < C'_x < \pi$. Moreover, an expression for t^*, the critical value of the lead time in case 2, can be obtained by setting $\pi = C'(t)$ on the RHS of (15); this process yields

$$t^* = -(1/B)\ln[(\pi - C'_n)/(C'_x - C'_n)]. \qquad (16)$$

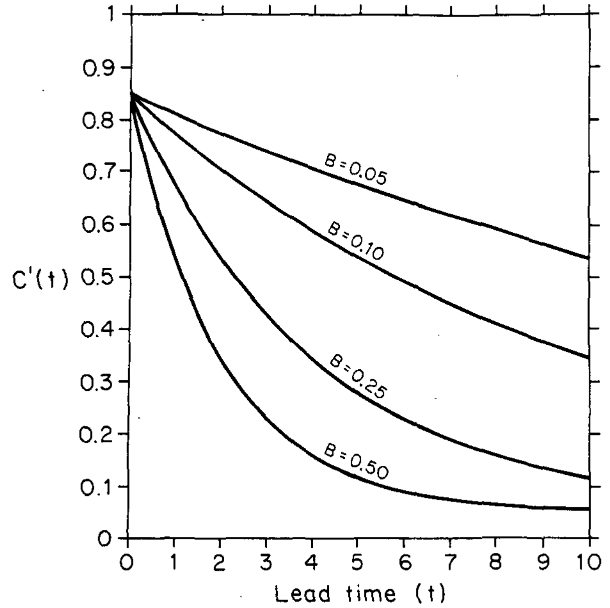

FIG. 4. The behavior of the exponential model of the cost parameter $C'(t)$ for selected values of the parameter B, with $C'_x = 0.85$ and $C'_n = 0.05$ [see (14)].

Examination of (16) reveals that t^* decreases (increases) as B (or C'_x or C'_n) increases (note that t^* is undefined and negative in cases 1 and 3, respectively). It should also be noted that t^* depends only on the parameters associated with the model for $C'(t)$ and the climatological probability (π).

The expression for $E'_C(t)$ in (15) is depicted in Fig. 5 for selected values of the parameter B, when $C'_x = 0.85$, $C'_n = 0.05$, and $\pi = 0.2$. These situations represent particular realizations of case 2, since $C'_n < \pi < C'_x$. Thus, $E'_C(t)$ is constant (0.2) for $t < t^*$, and it is a decreasing function of t for $t > t^*$. For these parameter values, $t^* = 1.67$, 2.09, and 3.35 when $B = 1$, 0.8, and 0.5, respectively.

2) Imperfect forecasts

For the exponential models of $p_1(t)$ and $C'(t)$ in (13) and (14), respectively, $E'_F(t)$ in (9) becomes

$$E'_F(t) = \pi\{1 + [(C'_x - C'_n)e^{-Bt} + C'_n] - [(1 - \pi)e^{-At} + \pi]\}. \quad (17)$$

As indicated in section 2c, minimizing expected expense when the imperfect forecasts are available to the decision maker involves finding $t_{\min}$, the value of t that minimizes $E'_F(t)$. This value of t can be found by differentiating $E'_F(t)$ in (17) with respect to t and setting the partial derivative equal to zero, which yields

$$t_{\min} = [1/(B - A)] \ln[B(C'_x - C'_n)/A(1 - \pi)]. \quad (18)$$

Examination of the second derivative of $E'_F(t)$ with respect to t reveals that $E'_F(t_{\min})$ is indeed a minimum if $B > A$. Thus, $E'_F(t)$ attains a minimum value only if the cost of protection decreases more rapidly (as lead time increases) than the accuracy of the forecasts. Moreover, $t_{\min} > 0$ if $B(C'_x - C'_n) > A(1 - \pi)$ and $t_{\min} < 0$ if $B(C'_x - C'_n) < A(1 - \pi)$. In the latter case, the realizable minimum value of $E'_F(t)$ actually occurs at $t = 0$. Therefore, to minimize expected expense in this case, the decision maker should wait until the last minute before making his/her decision. With regard to the sensitivity of $t_{\min}$ to the various parameters, it increases (decreases) as C'_x (C'_n) increases in all cases.

The expression for $E'_F(t)$ in (17) is depicted in Fig. 6 for selected values of the parameter A, when $B = 1$, $C'_x = 0.85$, $C'_n = 0.05$, and $\pi = 0.2$. For these sets of parameter values, $t_{\min} > 0$ [see (18)]. Moreover, the behavior of $E'_F(t)$ is similar in each case; it decreases initially (that is, for small values of t), reaches a minimum value at $t_{\min}$, and then increases for larger values of t. Specifically, $t_{\min} = 1.72$, 1.39, and 1.19 for $A = 0.3$, 0.5, and 0.7, respectively [from (18)]. As expected, $E'_F(t)$ decreases and $t_{\min}$ increases as A decreases (that is, as the rate of decrease in accuracy with increasing lead time decreases).

Note that $t_{\min}$ in (18) depends on the parameters associated with both the cost model and the accuracy model [that is, with $C'(t)$ and $p_1(t)$]. To investigate the relative sensitivity of $t_{\min}$ to changes in the values of the parameters A and B, we will let $\alpha = B/A$ and assume that both α and $D = \ln[\alpha(C'_x - C'_n)/(1 - \pi)]$ are constant. Then $\partial t_{\min}/\partial A = -D/A^2(\alpha - 1)$ and $\partial t_{\min}/\partial B = -\alpha D/B^2(\alpha - 1)$ [from (18)]. Thus, $(\partial t_{\min}/\partial A)/(\partial t_{\min}/\partial B) = \alpha$ and, since $t_{\min}$ exists only when $B > A$, this minimum lead time is more sensitive to changes in A (the accuracy parameter) than to changes in B (the cost parameter).

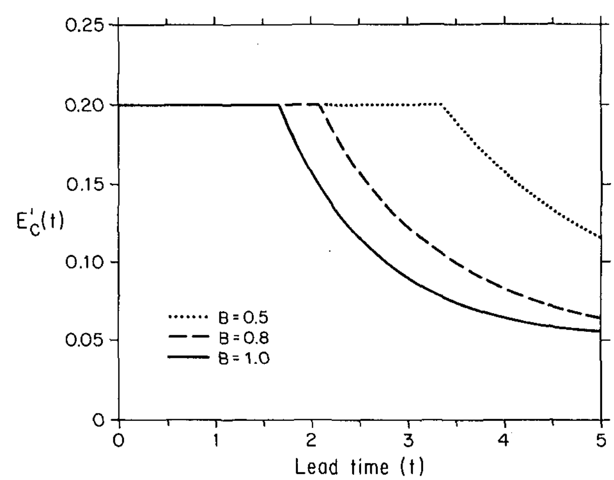

FIG. 5. The expected expense for climatological information, $E'_C(t)$, for selected values of the parameter B, with $C'_x = 0.85$, $C'_n = 0.05$, and $\pi = 0.2$ (case 2).

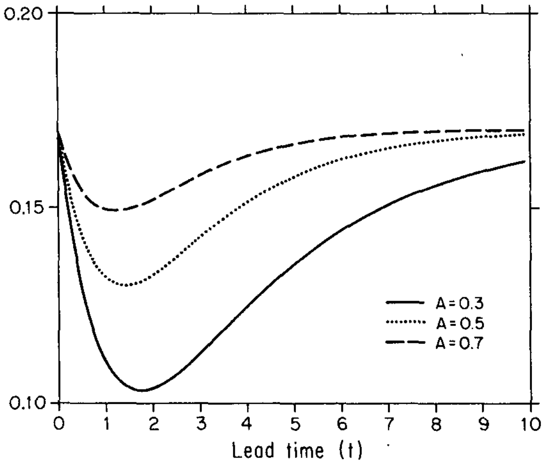

FIG. 6. The expected expense for imperfect forecasts, $E'_F(t)$, for selected values of the parameter A with $B = 1$, $C'_x = 0.85$, $C'_n = 0.05$, and $\pi = 0.2$ (case 2).

The behavior of $t_{\min}$ as a function of α ($=B/A$) for selected values of A and B is depicted in Figs. 7a and 7b, respectively. For fixed values of the accuracy parameter A (Fig. 7a), $t_{\min}$ decreases as α increases (that is, as B increases). This result is not surprising since increases in the parameter B imply that the cost of protection is decreasing more rapidly as lead time increases. Moreover, $t_{\min}$ is more sensitive to changes in α for small values of A (accuracy decreasing slowly) than for large values of A (accuracy decreasing rapidly). As expected, $t_{\min}$ is greater for small values of A than for large values of A.

For fixed values of the cost parameter B (Fig. 7b), $t_{\min}$ increases as α increases (since the accuracy parameter A is decreasing). In this case, $t_{\min}$ is more sensitive to changes in α for small values of B (cost decreasing slowly) than for large values of B (cost decreasing rapidly). Of course, $t_{\min}$ is greater for small values of B than for large values of B.

c. *Value of information*

For the exponential models of $p_1(t)$ and $C'(t)$ defined in (13) and (14), respectively, $V'_F(t)$in (12) becomes

$$V'_F(t) = \pi[(1-\pi)e^{-At} + \pi - (C'_x - C'_n)e^{-Bt} - C'_n], \quad \pi \leq C'(t) \quad (19a)$$

and

$$V'_F(t) = (1-\pi)[(C'_x - C'_n)e^{-Bt} + C'_n - \pi - \pi e^{-At}], \quad \pi > C'(t). \quad (19b)$$

Note that (19a) applies to the situation in which $\pi < C'(t)$ (the optimal action based on climatological information is *not* to protect) and (19b) applies to the situation in which $\pi > C'(t)$ (the optimal action based on climatological information is to protect). From the meteorologist's point of view, it is of interest to determine $t_{\max}$, the lead time at which $V'_F(t)$ in (19) is maximized. Differentiation of $V'_F(t)$ in (19a) with respect to t yields

$$t_{\max} = [1/(B-A)] \ln[B(C'_x - C'_n)/A(1-\pi)], \quad \pi < C'(t), \quad (20)$$

where $V'_F(t_{\max})$ is indeed a maximum if $B > A$. Comparison of (18) and (20) reveals that $t_{\max} = t_{\min}$, a result that is not surprising since $V'_F(t)$ and $E'_F(t)$ are linearly related in this situation [that is, when $E'_C(t) = \pi$; see (10)]. Thus, when the optimal action under climatological information is not to protect [$\pi < C'(t)$], the decision maker's optimal lead time $t_{\min}$ (minimum expected expense for imperfect forecasts) is identical to the forecaster's optimal lead time $t_{\max}$ (maximum expected value for imperfect forecasts).

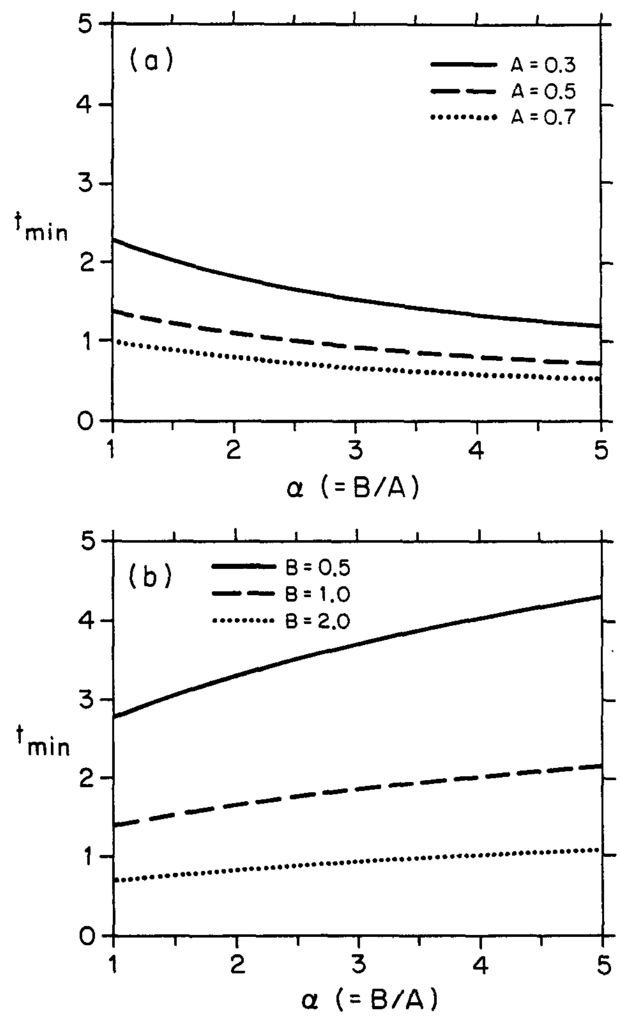

FIG. 7. The behavior of $t_{\min}$ as a function of α ($=B/A$) for selected values of (a) the accuracy parameter A and (b) the cost parameter B.

Examination of $V'_F(t)$ in (19b) [$\pi > C'(t)$] reveals that t_{max} is negative and that forecast value is a decreasing function of lead time for $t \geqslant t^*$ [the first derivative of $V'_F(t)$ is negative]. Therefore, the maximum value of $V'_F(t)$ in this situation must occur at the time at which $\pi = C'(t)$, which is by definition t^* (see section 3c). Thus, under the condition $\pi > C'(t)$, $t_{max} = t^*$ and, unless $t_{min} = t^*$ as well [an unlikely event; cf. (16) and (18)], $t_{max} \neq t_{min}$. In this situation, then, the decision maker's optimal lead time differs from the forecaster's optimal lead time.

With regard to the three cases introduced in section 2c (see also section 3b), recall that $E'_C(t) = \pi$ (a constant) in case 1 ($\pi < C'_n < C'_x$) for all values of t and in case 2 ($C'_n < \pi < C'_x$) when $t < t^*$. Obviously, $V'_F(t)$ is linearly related to $E'_F(t)$ in such situations. Thus, $t_{max} = t_{min}$ and $V'_F(t_{max}) = V'_F(t_{min})$ in these two situations.

On the other hand, $E'_C(t) = C'(t)$ (a decreasing function of t) in case 2 when $t > t^*$ and in case 3 ($C'_n < C'_x < \pi$) for all values of t. In both situations, $V'_F(t) = (1 - \pi)C'(t) - \pi[1 - p_1(t)]$ [see (12)]. Thus, $V'_F(t)$ is a decreasing function of t in these situations. In case 3, $t_{max} = 0$ and $V'_F(t_{max}) = V'_F(0)$. Therefore, the value of information would be maximized if the decision maker waited until $t = 0$ (zero lead time) to make his/her decision. On the other hand, $t_{min} \geqslant 0$ in this case, so that minimizing expected expense (the decision maker's *true* objective) and maximizing expected value may not be "consistent" objectives.

In case 2 when $t > t^*$, $t_{max} = t^*$ and $V'_F(t_{max}) = V'_F(t^*)$. Since $t_{min} \geqslant t^*$ in this case, it follows that $t_{min} \geqslant t_{max}$ and $V'_F(t_{max}) \geqslant V'_F(t_{min})$. Once again, the decision maker's and forecaster's objectives generally will not be consistent (in the sense of these objectives being realized at the same lead time). In particular, the realizable value of information [$V'_F(t_{min})$] from the decision maker's viewpoint will generally be less than the potential value of information [$V'_F(t_{max})$] from the forecaster's viewpoint.

4. Some numerical results

This section presents numerical examples based on the time-dependent model of the cost–loss ratio situation described in section 2b and the exponential accuracy and cost-of-protection models defined in section 3a. Examples are considered for each of the three cases identified in section 2c (see also section 3b). In choosing parameter values for these examples, we have assumed that the units of lead time (t) are "days," and we have specified values of the accuracy parameter (A) that are consistent with the current state-of-the-art of day-to-day weather forecasting.

a. Example for case 1

As a numerical example for case 1 ($\pi < C'_n < C'_x$), consider a situation in which $C'_x = 0.85$, $C'_n = 0.25$, and $\pi = 0.2$ (that is, the cost of protection ranges from 25% of the loss L at very long lead times to 85% of L at very short lead times and the climatological probability of adverse weather is 0.2). In this case, the optimal action with climatological information is not to protect for all values of t[$\pi < C'(t)$ or, more specifically, $\pi < C'_n$]. Thus, the imperfect forecasts can be of positive value only if it is optimal, at least occasionally, for the decision maker to take protective action.

Figure 8 depicts $E'_C(t)$, $E'_F(t)$, and $V'_F(t)$ in this example when $A = 0.5$ and $B = 1$. For $t \leqslant 0.14$, $E'_F(t) = E'_C(t) = 0.2$ and (consequently) $V'_F(t) = 0$. Evidently, for these very short lead times, even very accurate forecasts cannot "offset" the (relatively) substantial cost of protection. In fact, for these values of t, $p_1(t) \leqslant C'(t)$.

For larger values of $t (t \geqslant 0.14)$, $E'_F(t)$ decreases to a minimum at t_{min} and then increases once again. Since $E'_C(t)$ is a constant (π) for all t, $V'_F(t)$ is the mirror image of $E'_F(t)$ and $t_{max} = t_{min}$. For the specified parameter values, $t_{min} = t_{max} = 1.40$, $E'_F(t_{min}) = 0.170$, and $V'_F(t_{max}) = V'_F(t_{min}) = 0.030$. In this case, an intermediate lead time exists at which it is optimal for the decision maker to make his/her decision.

b. Examples for case 2

Two numerical examples are considered for case 2 ($C'_n < \pi < C'_x$), both of which involve a situation in which $C'_x = 0.85$, $C'_n = 0.05$, and $\pi = 0.2$. The behavior of $E'_C(t)$ and $E'_F(t)$ in this situation was described in Figs. 5 and 6, respectively. In the first example, $E'_C(t)$, $E'_F(t)$, and $V'_F(t)$ are depicted in Fig. 9a for the particular situation in which $A = 0.5$ and $B = 1$. Here, $t_{min} = 1.39 < t^* = 1.67$, so that $t_{max} = t_{min}$ and $V'_F(t_{max})$

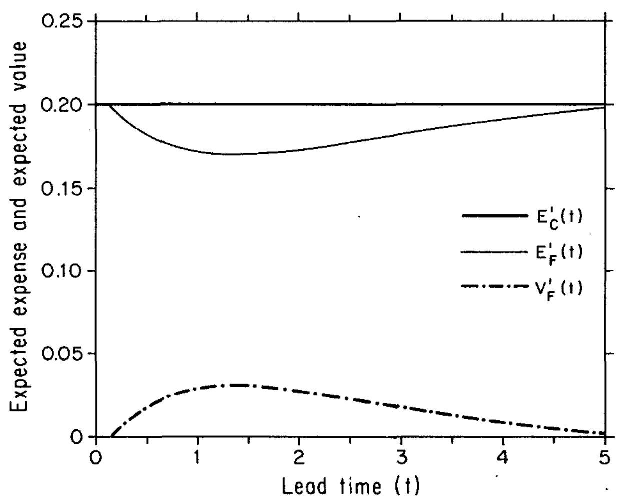

FIG. 8. $E'_C(t)$, $E'_F(t)$, and $V'_F(t)$ for a numerical example in case 1 ($\pi < C'_n < C'_x$) for which $A = 0.5$, $B = 1$, $C'_x = 0.85$, $C'_n = 0.25$, and $\pi = 0.2$.

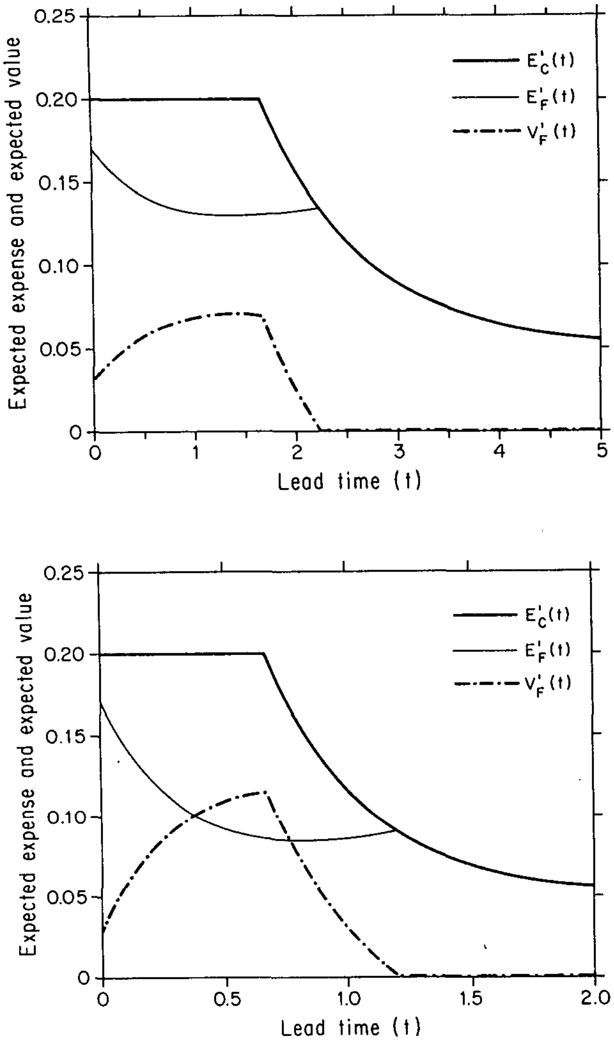

FIG. 9. $E'_C(t)$, $E'_F(t)$, and $V'_F(t)$ for numerical examples in case 2 ($C'_n < \pi < C'_x$) for which (a) A = 0.5, B = 1, C'_x = 0.85, C'_n = 0.05, and π = 0.2 and (b) A = 0.5, B = 2.5, C'_x = 0.85, C'_n = 0.05, and π = 0.2.

= $V'_F(t_{\min})$. In particular, $E'_C(t_{\min}) = \pi = 0.2$, $E'_F(t_{\min})$ = 0.132, and $V'_F(t_{\min})$ = 0.068. It should be noted that, in the vicinity of $t_{\min}$, both $E'_F(t)$ and $V'_F(t)$ are relatively flat functions; that is, the values of these functions are relatively insensitive to small changes in t.

In the second example, $E'_C(t)$, $E'_F(t)$, and $V'_F(t)$ are depicted in Fig. 9b for the situation in which A = 0.5 and B = 2.5. Here, $t_{\min} = 0.81 > t^* = t_{\max} = 0.65$. Moreover, $E'_C(t_{\min})$ = 0.158, $E'_F(t_{\min})$ = 0.084, and $V'_F(t_{\max}) = V'_F(t^*)$ = 0.114. It is of interest to compare the latter with $V'_F(t_{\min})$, where $V'_F(t_{\min})$ = 0.074. Thus, in this example, the decision maker's objective of minimizing expected expense leads to a value-of-information estimate that is considerably less than the maximum value-of-information estimate. Once again, however, $E'_F(t)$ is a relatively flat function in the vicinity of $t_{\min}$ (=0.81). In fact, $E'_F(t_{\max})$ is only 0.086.

c. *Example for case 3*

As a numerical example for case 3 ($C'_n < C'_x < \pi$), we consider a situation in which C'_x = 0.15, C'_n = 0.05, and π = 0.2. In this case, the optimal action with climatological information is to protect for all values of $t[\pi > C'(t)$ or, more specifically, $\pi > C'_x]$. Thus, the imperfect forecasts can be of value only if it is optimal for the decision maker not to take protective action for some forecasts.

Figure 10 depicts $E'_C(t)$, $E'_F(t)$, and $V'_F(t)$ in this situation when A = 0.5 and B = 1. For these parameter values, $E'_F(t)$ and $V'_F(t)$ are increasing and decreasing functions of t, respectively. Thus, the minimum (maximum) value of $E'_F(t)[V'_F(t)]$ occurs at t = 0. Specifically, $E'_C(0)$ = 0.15, $E'_F(0)$ = 0.03, and $V'_F(0)$ = 0.12. Moreover, $V'_F(t)$ = 0 for $t \geqslant 1.086$. In this example, the decision maker evidently should wait as long as possible before making his/her decision, since the cost of protection is relatively small and the decision not to protect is likely to be "beneficial" only when the forecasts are relatively accurate. Moreover, the decision maker's and forecaster's objectives can both be realized at the same lead time (at least for the parameter values employed in this example).

5. Discussion and conclusion

In this paper we have described a time-dependent version of the cost–loss ratio situation. Motivation for considering such a situation was provided by a decision maker who can postpone the protect/do not protect decision, but who faces a potential tradeoff as lead time decreases between increasingly more accurate forecasts and increasingly more costly protective measures. Exponential models of the accuracy and cost-of-protection

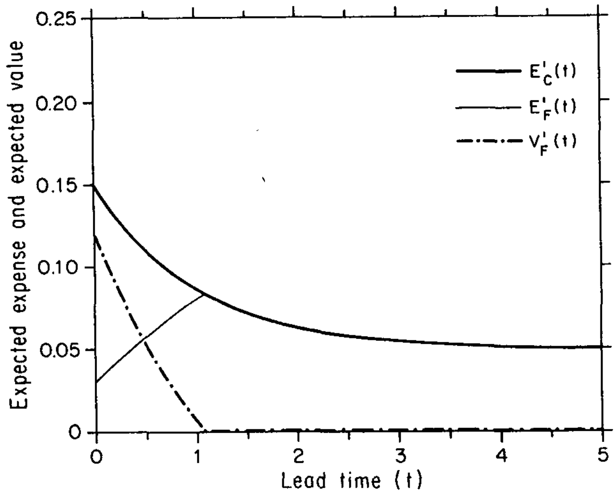

FIG. 10. $E'_C(t)$, $E'_F(t)$, and $V'_F(t)$ for a numerical example in case 3 ($C'_n < C'_x < \pi$) for which A = 0.5, B = 1, C'_x = 0.15, C'_n = 0.05, and π = 0.2.

parameters were employed as a means of investigating the optimal use and economic value of calibrated (but otherwise imperfect) categorical forecasts in this context. Expressions for the expected expense and economic value of the forecasts were obtained for these models, leading to the identification of optimal lead times for minimizing expected expense and maximizing expected (forecast) value. The dependence of the results (minimum expected expense, maximum economic value, optimal lead time) on the values of various model parameters was investigated and these results were illustrated by means of numerical examples.

Two general conclusions can be drawn from this work. First, under the assumption that time-dependent situations such as that considered here actually exist in the real world, the results of this study demonstrate that it is beneficial in at least some circumstances for an individual faced with such a decision-making problem to determine the optimal time at which to make his/her decision. That is, it may not be optimal to make the decision at the earliest opportunity or to postpone the decision until the last minute; in some cases, the decision should be made at an *intermediate* lead time. Of course, the optimal lead time in any particular situation will depend (*inter alia*) on the manner in which forecast quality and the decision maker's "payoff structure" (that is, costs and losses) change as a function of lead time.

Another important conclusion arises from the fact that the optimal lead time from the decision maker's viewpoint ($t_{\min}$, the time at which expected expense is minimized) does not always correspond to the optimal lead time from the forecaster's viewpoint ($t_{\max}$, the time at which forecast value is maximized). The results presented in sections 3 and 4 reveal that $t_{\max} = t_{\min}$ in some situations (that is, when the optimal action for climatological information is "do not protect"), but that $t_{\max} \neq t_{\min}$ in other situations (that is, when the optimal action for climatological information is "protect"). Of course, the decision maker's role is primary in this context, since forecasts acquire value only through their use (Winkler and Murphy 1985). In situations in which $t_{\max} \neq t_{\min}$, the forecaster can still take considerable satisfaction from the fact that his/her forecasts have enabled the decision maker to minimize expected expense, even though these forecasts may not always achieve the maximum possible economic value.

With regard to conclusions of a more specific nature, it is of interest to consider briefly the implications of certain results insofar as they relate to the parameters of the exponential models. Recall that an optimal lead time for minimizing expected expense exists only if the magnitude of the parameter B (which characterizes the rate of increase in the cost of protection as lead time decreases) exceeds the magnitude of the parameter A (which characterizes the rate of increase in forecast accuracy as lead time decreases). This condition suggests that the exponential model may be particularly appropriate in those situations in which the cost of protection increases quite rapidly as lead time decreases and is relatively high for very short lead times. The identification of real-world situations that satisfy these conditions would be a useful endeavor. In a related vein, it should be noted that increases in forecast accuracy (associated with improvements in the state-of-the-art of weather/climate forecasting) could lead to decreases in the value of the parameter A, thereby increasing the likelihood that this condition ($B > A$) would be satisfied for a particular decision maker. Thus, improvements in forecast quality would not only reduce the expected expense and increase the optimal lead time for decision makers for whom this condition already held, but they would also increase the number of decision makers who could potentially realize such benefits.

The sensitivity of the results to the climatological probability (π) is also of interest. Examination of the expression for the optimal lead time [$t_{\min}$; see (18)] indicates that $t_{\min}$ increases (decreases) as π increases (decreases). Thus, for relatively rare events (π small), which generally also represent adverse weather/climate conditions, the optimal lead time is relatively short. On the other hand, for relatively frequent events (π large) the optimal lead time is relatively long.

Although we believe that the results presented in this paper provide useful initial insights into optimal decision making and forecast value in time-dependent situations, many possible extensions of this work can be identified. First, in the context of the cost–loss ratio situation, it would be interesting to investigate forecast use and value for other time-dependent models of accuracy and cost of protection. In addition, it would be useful to relax the condition that the marginal probability of a forecast of adverse weather is equal to the climatological probability of adverse weather ($p_z = \pi$; see section 2a), as well as to explore the use and value of (multivalued) probability forecasts in this context. Another possible extension would involve incorporating the decision maker's attitude toward risk, especially since his/her risk preferences might not be constant over the range of relevant lead times (as assumed here). Finally, it would be desirable to explore optimal decision making and forecast value in time-dependent problems of greater complexity and/or dimensionality; that is, problems involving more than two actions and two events and including more realistic treatments of the respective payoff and information structures. Such studies would further enrich the class of prototype decision-making models that are currently available to investigate the use and value of weather/climate forecasts.

Acknowledgments. Richard W. Katz provided useful comments on an earlier version of this paper. This work was supported in part by the National Science Foundation (Division of Atmospheric Sciences) under Grant ATM-8714108.

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